

# Fault-tolerance against loss for photonic FTQEC

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In general, fault-tolerant quantum error correction (FTQEC) procedures are designed to detect, correct, and be fault-tolerant against errors occurring within the qubit subspace. But in some qubit implementations, additional “leakage” errors can occur in which the system leaves this subspace, and standard FTQEC procedures may not be fault-tolerant against such errors. Generic methods for achieving fault-tolerance against leakage are costly in terms of resources.

In this paper we demonstrate that for a leakage model common to many photonic gate implementations, FTQEC can be implemented with far fewer additional operations than in the generic case.

## I. INTRODUCTION

Physical systems used for the storage, transport or processing of quantum information are vulnerable to noise introduced through interaction with the environment. One effective means of preserving a quantum system against such noise is the use of quantum error-correcting codes (QECCs), which encode quantum information into larger physical systems (e.g., encoding one qubit into multiple qubits) such that certain types of error can be correctly identified and corrected without damaging the encoded state in the process. These typically have properties denoted by  $[[n, k, d]]$ : a code encoding  $k$  qubits into  $n$  with code distance  $d$ , corresponding to an ability to correct an arbitrary error on  $(d-1)/2$  of the  $n$  qubits.

While the necessary operations for quantum error correction (QEC) can be implemented using standard physical quantum gates, any realistic implementation must take into account that these gates themselves will be imperfect, and the QEC process can both introduce as well as correct errors. This leads to the requirement that for effective QEC using imperfect components, the QEC must be *fault-tolerant*, i.e., it should be constructed such that a single gate failure in the QEC cannot lead to errors on multiple data qubits. Many different QECCs with fault-tolerant constructions are known.

QEC analysis typically considers quantum information encoded into two-level systems (qubits), where the QECC can correct some set of data errors which manifest as state changes within the qubit subspace of the data qubits and/or any ancillary qubits used (for which it is known to be sufficient to consider only combinations of Pauli  $X$  (bit-flip) and  $Z$  (phase-flip) operations). We will refer to these errors, occurring within the subspace, as “standard” errors. However, real physical implementations will, in general, also be subject to errors which take the system outside of the qubit subspace. For example, the qubit may be represented by two energy levels of a trapped ion, or orthogonal polarizations of a photon, but it may also undergo interactions which take it

to a third energy level, or change the photon’s position. These are typically referred to as “leakage” errors. A QECC which can correct, and is fault-tolerant against, qubit errors, may not have these properties with respect to leakage errors, and a gate failure within such a QECC which causes a leakage error may lead to multiple errors (standard or leakage) on the data.

One of the most common examples of leakage is the loss of photons in photonic systems. Photonic qubits are typically encoded via timing, polarisation or positional (dual-rail) encodings (with the latter two freely interchangeable through polarising beam-splitters (PBSs)). However, most optical components have some non-zero probability for loss of a photon through absorption or scattering out of the optical system. This is not equivalent to a qubit error (e.g. absorption of a photon does not correspond to any change of polarisation), and the probability of such errors can be comparable to or even larger than those of standard errors.

Thus, in general, QEC procedures must be modified to correct and be fault-tolerant against leakage. A comprehensive procedure for achieving fault-tolerance in a general leakage model was previously given by Aliferis and Terhal [1]. This was based on the use of “leakage replacement units” (LRUs), devices that would leave non-leaked qubits unchanged, but replace any leaked qubit with a qubit in some arbitrary state, thus converting a leakage error into (at worst) a standard error on that qubit. Various forms of LRU can be constructed; one example would be a standard teleportation circuit. For a given circuit, however, large numbers of LRUs may be required to achieve fault-tolerance against leakage errors, which can impose significant additional resource costs (primarily in terms of additional gates, rather than time) to the computation.

In this paper we demonstrate that, for certain FTQEC procedures, these costs may be substantially reduced for certain general models of qubit loss. In section II we specify the behaviour of our model with respect to losses. In section III we show our main result of how the existing

Steane, Shor and Knill ancilla techniques for distance-3 CSS codes may be minimally modified to achieve fault-tolerance against losses which follow this model. Finally in section IV we show how our analysis is applicable to QEC in higher-distance codes.

## II. LOSS MODEL

As discussed above, since loss errors fall outside of the qubit space, to understand the behaviour of a QEC circuit in the presence of loss errors requires the description of the quantum gate behaviour to be supplemented with a description of its behaviour in the presence of loss. We consider a model which, in addition to being subject to standard errors, satisfies the following behaviour:

- Every gate (including identity/memory operations and LRUs) may undergo a fault in which the qubit(s) it acts on becomes lost (thus a two-qubit gate can lose one or both qubits it acts on).
- A qubit, once lost, remains lost regardless of any further operations it may undergo, with the exception of LRUs, which when correctly operating will replace the lost qubit with one which may carry a qubit error.
- **A two-qubit operation in which one input qubit is lost will (unless an additional fault occurs) perform the identity operation on the non-lost qubit.**

We note that the above does not describe all behaviour with respect to lost qubits (e.g. what happens when a lost qubit is measured) but is sufficient to demonstrate the resource reduction presented here, regardless of the unspecified behaviour. The feature which is distinct to our model is the behaviour of the two-qubit operations. As discussed above, one physical implementation very subject to qubit losses is the use of photonic qubits. The above model is applicable to the large number of proposed photonic 2-qubit gates [2], in which the input photonic qubits are in the dual-rail basis, and a phase shift  $\theta$  occurs if and only if both input photons are in state  $|1\rangle$  and hence undergo the appropriate interaction, as depicted in Figure 1. Possible means of implementing such a phase shift include direct use of a Kerr nonlinear medium [3], mediation via an atom in cavity QED [4], the use of the Zeno effect [5] or electromagnetically-induced transparency [6]. The details do not matter to our model, so long as the gate satisfies the behaviour described above. Such a gate, in the dual-rail basis, performs the operation

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \rightarrow \{|00\rangle, |01\rangle, |10\rangle, e^{i\theta}|11\rangle\}. \quad (1)$$

Additional PBSs may be added before and after the gate to convert it to operation in the polarisation basis, as shown in the dashed boxes in Figure 1. For  $\theta = \pi$  this

corresponds to the standard controlled-phase gate, and in the polarisation basis this can be simply converted to a CNOT gate i.e.

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \rightarrow \{|00\rangle, |01\rangle, |11\rangle, |10\rangle\}. \quad (2)$$

by adding Hadamard gates in the form of half-wave plates (HWPs) applied to the target photon for the CNOT before and after the phase shift gate.

It is clear that a gate working in this manner will behave according to our model: to perform as described when both qubits are present, the state of either input qubit will change if and only if the other qubit is present in the  $|1\rangle$  channel. Thus if either input qubit is lost, the state of the other qubit will remain unchanged.

### A. Measurement

As mentioned above, our model does not specify the behaviour of measurement operations on lost qubits. However, we can assume that the measurement operations we consider (projective measurements on a qubit) have two outcomes [7] and that a measured lost qubit will therefore produce one of the two outcomes with some probability, independently of other qubits.

To show, with respect to fault-tolerance, that this behaviour is equivalent (at worst) to the qubit having a standard error, we need to only show that either measurement outcome, combined with any behaviour we are considering, could occur from a qubit with a standard error (or from no error), without any additional errors occurring. Note that the specific probabilities of the measurement outcomes do not have to be the same in the loss and standard error cases.

As an example (which will occur in the cases we consider), suppose we wish to show that within a procedure a loss error is equivalent to a replacement by a qubit in state  $|0\rangle$  (a standard error), for a particular qubit which is then measured in the  $X$  basis. The standard error will then produce measurement outcome 0 or 1 at random (independently of other qubits) with equal probabilities, therefore if the procedure is fault-tolerant against standard errors neither of these outcomes can lead to multiple data errors. Hence regardless of the specific behaviour of the measurement with respect to loss (what the probabilities of 0 or 1 are) the procedure will still be fault-tolerant in the presence of the loss error.

## III. THE STEANE, SHOR AND KNILL QEC TECHNIQUES

Many QECCs use one of three general ancilla techniques for QEC, which we will describe here as the Steane, Shor and Knill techniques, and consider with regard to their fault-tolerance against loss.

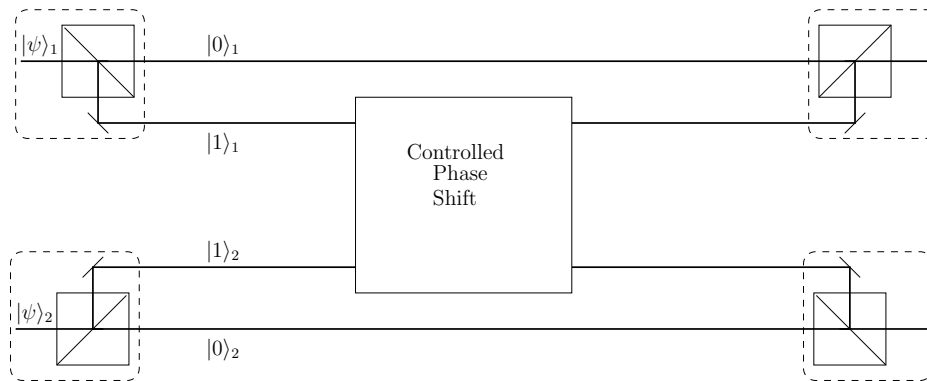


FIG. 1. A 2-qubit phase gate satisfying our model with regard to loss errors.

### A. Steane QEC

Error-correction of  $Z$  errors in Steane QEC [8] consists of the following

- An encoded ancilla state  $|\bar{0}\rangle$  is prepared.
- A transversal CNOT gate is performed with the ancilla as the control and the data as the target. This leaves the state unchanged at the logical level but any  $Z$  errors on the data are transferred to the ancilla.
- The ancilla is then measured transversally in the  $X$  basis and the error syndrome calculated. This will be the syndrome corresponding to the  $Z$  errors on the ancilla, those acquired from the data plus any others the ancilla may have had.
- The data is then corrected according to this error syndrome.

Essentially the same procedure applies for correcting  $X$  errors, except with preparation of a  $|\bar{\tau}\rangle$  ancilla, which is used as the control in a CNOT with the data as the target, and then measured in the  $Z$  basis.

For a distance-3 code, fault tolerance will be violated if and only if a single gate failure leads to a logical error on the data. The only operations performed directly on the data are the (transversal) CNOT and (if necessary) the correction operation, which will affect 1 data qubit per fault at most. Thus no single gate failure on these operations can lead to multiple data errors. Similarly, no set of errors on the ancilla which occurs after it interacts with the data can violate fault-tolerance, since these only affect the syndrome measurement and hence the (single-qubit) correction operation.

The only remaining possible sources of violation are gate failures in either the ancilla preparation or the ancilla interaction with the data (i.e., the CNOT gates). Since in the Steane QEC the ancilla measurement occurs immediately after the CNOT interaction with the data, a CNOT failure in this operation leading to an ancilla qubit loss is equivalent to a failure leading to a

standard ancilla qubit error (since both will, in general, cause measurement errors), against which the protocol is already fault-tolerant. If the CNOT failure causes losses in both data and ancilla qubits, any incorrect syndrome due to the lost ancilla qubit will simply result in a correction operation on the (corresponding) lost data qubit, leading to only a single data loss error.

We are left with the case where a gate failure occurs in the ancilla preparation, leading to multiple loss and/or standard errors on the ancilla qubits prior to interaction with the data. This is a potential problem even in the absence of loss, hence the use of *ancilla verification* in preparation. For example, in the case of  $|\bar{0}\rangle$  ancillas, multiple  $X$  errors could be transferred to the data. Hence a second, identical ancilla is prepared (the “verifier”), and used as the target in a transversal CNOT from the original ancilla, then measured in the  $Z$  basis. Any  $X$  errors from the original ancilla are transferred to the verifier resulting in an error syndrome and/or logical  $X$  operation on the verifier, which if detected cause the ancilla to be rejected and a new one prepared. Ancillas must pass verification before interaction with the data (an analogous process occurs with the  $|\bar{\tau}\rangle$ ), and this can be shown to be fault-tolerant against standard errors.

More specifically, no pattern of standard qubit errors on either the newly-created ancilla or verifier *alone* (regardless, in fact, of whether the pattern can be produced by a single gate failure) can result in a logical error on the data in the absence of additional errors. This can be seen since, in a CSS stabiliser code, any set of  $X$  operations on the qubits of an encoded state can be represented as a product of stabiliser operators, logical  $\bar{X}$  operations and correctable errors. Stabilisers can be ignored as they leave the state unchanged, and any combination of logical operations and correctable errors on the verifier will be detected in verification in the absence of additional errors. Thus such a pattern on the verifier alone will result in verification failing. Errors on the ancilla state alone will be correctly transferred to the verifier and hence also result in failure. (Errors on both states might not result in failure, but we need not consider this possibility to show fault-tolerance for distance-3). Since only  $X$  errors

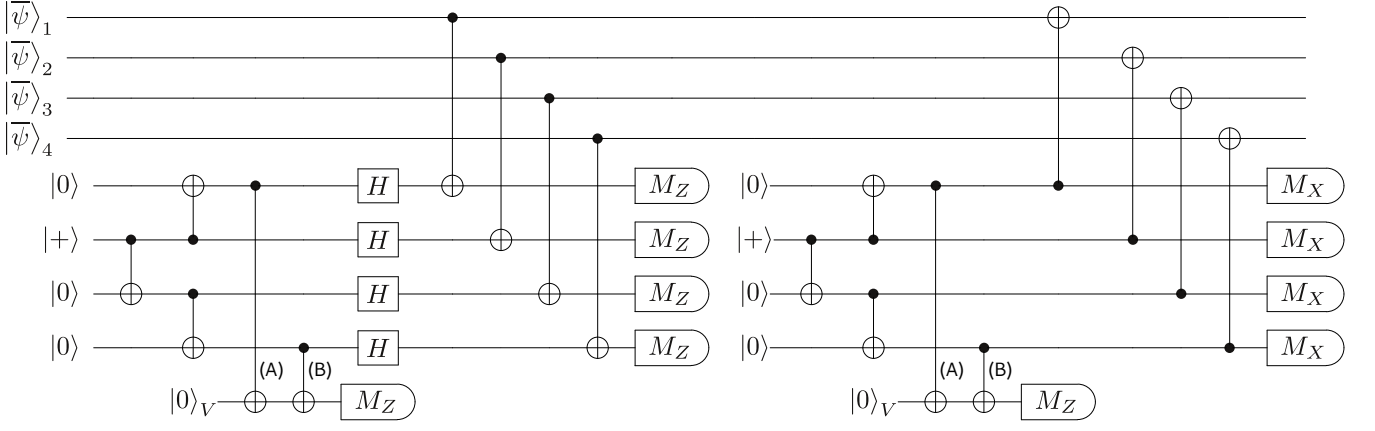


FIG. 2. Measurement of two stabilisers (one a product of  $X$  operators, one of  $Z$ ) on 4 qubits using the Shor technique in a  $[[7, 1, 3]]$  QECC.

can be transferred to the data in the  $Z$ -error correction circuit, this is sufficient to show fault-tolerance (an analogous case applies to  $Z$  errors in  $X$ -correction) and moreover that fault-tolerance applies for any *arbitrary* state (without losses) of the ancilla or verifier after encoding, since in standard QEC it is sufficient to consider  $X$  and  $Z$  errors alone.

We now consider the case of ancilla and/or verifier losses. The main observation our analysis depends on is that **in our loss model, the behaviour of a lost control/target qubit in a CNOT gate (i.e., having no effect on the remaining qubit) is equivalent to its having been replaced by a qubit in state  $|0\rangle/|+\rangle$** . Since such a replacement is a standard single qubit error, if we can demonstrate that a qubit loss is equivalent to this replacement in all subsequent interactions, then a standard fault-tolerant QEC (minimally modified so that any losses on data qubits are replaced) should also be tolerant against this loss. This is illustrated in Figure 3. Note that this is not always the case; for example, a qubit which is lost and subsequently acts as both control and target in separate CNOT gates would have to be modelled as undergoing at least two replacements, thus two separate qubit errors, which may not be a tolerable fault, as shown in Figure 4.

Consider then a newly-created  $|\bar{0}\rangle$  ancilla. Every qubit in this ancilla undergoes three subsequent interactions: as a CNOT control interacting with the verifier, CNOT control interacting with the data, and measurement in the  $X$  basis. Suppose some subset of these qubits have been lost in the ancilla creation, resulting in some arbitrary state of the remaining ancilla qubits. The lost qubits, acting as controls in the two CNOT interactions, will behave just as though they had left the ancilla creation circuit in state  $|0\rangle$ . Upon being measured in the  $X$  basis their behaviour will be no more harmful than that of a  $|0\rangle$  state, which would give a result of 0 or 1 at random. Thus the overall ancilla state is equivalent to a state with a set of standard errors after creation, against

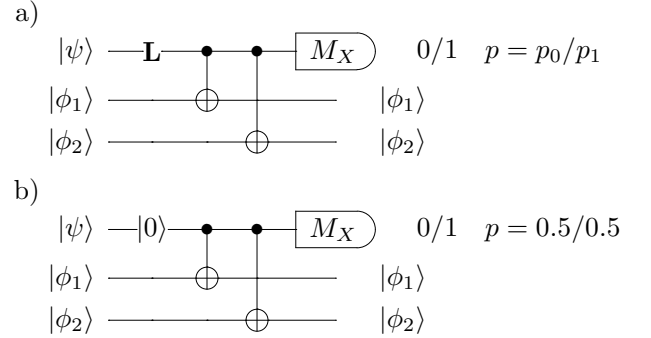


FIG. 3. A qubit a) undergoing a loss error  $\mathbf{L}$  can be equivalent in its behaviour to one which b) underwent the standard error of being replaced by state  $|0\rangle$ . In this example the one possible difference is the values of  $p_0$  and  $p_1$  in the measurement of the lost qubit (undefined in our model), but as discussed in Section IIA this does not matter with respect to fault-tolerance

which the QEC is already fault-tolerant. A similar argument applies to the verifier, which acts as a CNOT target only, followed by  $Z$  basis measurement, and whose losses are thus equivalent to replacement with  $|+\rangle$ . Analogous arguments apply to the  $|\bar{+}\rangle$  ancillas in  $X$  error correction. Thus overall the process is fault-tolerant, and to modify the QEC to cope with loss errors we need only add a set of LRUs on the data before the QEC. This is contrast to the more general model of [1], which also requires LRUs on all encoded ancilla states immediately after their creation.

## B. Shor QEC

In the Shor QEC technique [9], shown in Figure 2,  $n$ -qubit cat states  $|\text{cat}_n\rangle = |0\rangle^{\otimes n} + |1\rangle^{\otimes n}$  are used as ancillas. Unlike in the Steane technique, where measurement of the ancilla gives the complete error syndrome,

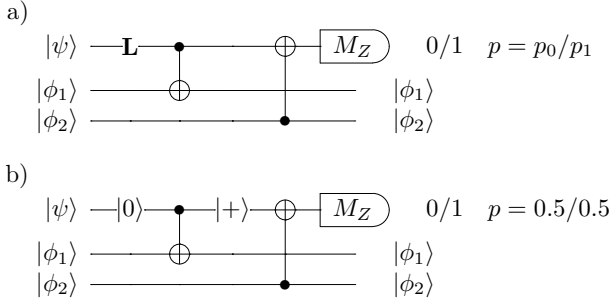


FIG. 4. If a qubit a) undergoes a loss error  $\mathbf{L}$  and subsequently act as both control and target in CNOT gates, then b) we may need to model its behaviour as multiple standard errors, in this case replacement by states  $|0\rangle$  and  $|+\rangle$  respectively.

the Shor technique provides the value of a single weight- $n$  stabiliser operator per ancilla, a product of either all  $X$  or all  $Z$  operations on individual qubits (as will be the case for stabilisers in a CSS code). The  $n$ -qubit stabiliser operation  $S_i$  is applied to the cat state conditioned on the state of the corresponding qubits on the data, by applying control- $X$  (i.e. CNOT) or control- $Z$  (i.e. CPHASE gates). Thus, for an  $X$  stabiliser  $S_i$  and even  $n$  we have

$$\begin{aligned} \text{CNOT}(|\text{cat}_n\rangle, |\text{data}\rangle) \\ = \frac{1}{\sqrt{2}} (|0\rangle^{\otimes n} |\text{data}\rangle + |1\rangle^{\otimes n} S_i |\text{data}\rangle) \end{aligned} \quad (3)$$

$$\begin{aligned} = \frac{1}{2^{\frac{n+1}{2}}} \left[ (|+\rangle + |-\rangle)^{\otimes n} |\text{data}\rangle \right. \\ \left. + (|+\rangle - |-\rangle)^{\otimes n} S_i |\text{data}\rangle \right] \end{aligned} \quad (4)$$

$$\begin{aligned} = \frac{1}{4} \left[ |\text{even parity}\rangle (|\text{data}\rangle + S_i |\text{data}\rangle) \right. \\ \left. + |\text{odd parity}\rangle (|\text{data}\rangle - S_i |\text{data}\rangle) \right] \end{aligned} \quad (5)$$

where e.g.  $|\text{even parity}\rangle$  is the equal superposition of  $n$ -fold tensor products of  $\{|+\rangle, |-\rangle\}$  containing an even number of  $|+\rangle$  states and similarly for  $|\text{odd parity}\rangle$ . As can be seen, the states  $\frac{|\text{data}\rangle \pm S_i |\text{data}\rangle}{\sqrt{2}}$  are the  $\pm 1$  eigenstates of the  $S_i$  operator, thus measurement of the parity of the ancilla state (by measuring individual ancilla qubits in the  $X$  basis) acts as a measurement of the stabiliser on the data, as required. (Similarly, for odd  $n$ , measurement of odd/even parity corresponds to  $+1/-1$  eigenstates of  $S_i$  respectively).

Various methods are applied to achieve fault-tolerance. The cat states are verified before interacting with the data, to prevent single gate failures in cat state preparation causing multiple errors which transfer to the data on interaction. Additionally, the complete syndrome is measured three times, and only used for error correction if at least two out of the three results agree (this prevents certain non-fault tolerant cases where the failure of the gate interacting the data and ancilla leads to both an error on the data and an incorrect syndrome which

causes a second error on correction). Given this feature, a single incorrect syndrome (which is the most a single gate failure can cause) cannot lead to any errors on the data.

Thus, in a model involving loss, the only remaining means by which a single gate failure can lead to multiple data errors is in the cat state preparation and verification. For either  $X$  or  $Z$  error correction, only  $X$  errors from the cat-state preparation can transfer to the data, thus we consider only these errors and losses. This leaves 5 qubit preparations and 5 CNOT gates as operations whose loss-related failures we need to investigate. In qubit preparation operations, we need only consider loss of the qubit. For the CNOT gates there are 5 possible relevant failure types involving loss:

- Control qubit is lost.
- Target qubit is lost.
- Both control and target qubits are lost.
- Control qubit is lost, target qubit has  $X$  error.
- Target qubit is lost, control qubit has  $X$  error.

We model errors occurring due to gate failure as taking place immediately after the successful operation of the gate in question (this is without loss of generality because, for example, an error modelled as occurring after a given gate can equivalently be modelled as occurring before the subsequent gate, thus consideration of all possible errors in one model will also cover all possible errors in another model, even if they are identified with different gates). We first consider failure of CNOT gates other than the final two (those which interact with the verifier, labelled (A) and (B) in Figure 2).

As seen from the circuit diagram, when preparing the cat state, one of the qubits is initialised in state  $|+\rangle$ , the others in state  $|0\rangle$ , and the cat state is progressively grown from states  $|0\rangle^{\otimes(m-1)} + |1\rangle^{\otimes(m-1)}$  to states  $|0\rangle^{\otimes m} + |1\rangle^{\otimes m}$  by using one of the most recently-added qubits from the  $(m-1)$ -qubit cat state as a source and an additional  $|0\rangle$  qubit as the target in a CNOT gate. If a qubit is lost from the cat state, the state will dephase to an equal mixture of  $|0\rangle^{\otimes(m-1)}$  and  $|1\rangle^{\otimes(m-1)}$  (which corresponds to applying an irrelevant  $Z$  error with 50% probability to one of the qubits) and any further cat state growth involving that qubit is halted, with additional qubits remaining in state  $|0\rangle$ . If a qubit is lost prior to becoming part of the cat state, no dephasing occurs, but cat state growth is similarly halted.

Thus, before interacting with the verifier, the ancilla qubit state will, in general, be a combination of a cat state of  $M$  qubits (coherent or dephased), either one or two lost qubits, and additional qubits in state  $|0\rangle$ . The two qubits which interact with the verifier are the final qubits added in two independent cat state growth processes; thus a single gate failure involving loss will cause one of them to be in state  $|0\rangle$  and the other to be part of





will also have two loss errors, corresponding to a logical error in many circumstances. We discuss in general terms (applicable to any CSS code) how this can occur below.

#### D. Passing verification with loss errors

Consider a single gate failure in preparation of a  $|\bar{0}\rangle_a$  state which has caused two qubit losses as discussed above, and suppose this occurs in one of the final gates in the preparation, such that the state of the remaining qubits is

$$\text{tr}_{AB}|\bar{0}\rangle\langle\bar{0}| = \sum_{\{i,j\} \in \{0,1\}} p_{ij}|\psi_{ij}\rangle\langle\psi_{ij}|_a \quad (6)$$

where  $A$  and  $B$  are the lost qubits, and

$$p_{ij}|\psi_{ij}\rangle\langle\psi_{ij}| = \langle ij|\bar{0}\rangle\langle\bar{0}|ij\rangle \quad (7)$$

Consider performing (without additional errors) a verification of this ancilla against  $X$  errors, by using it as the source and a  $|\bar{0}\rangle_v$  verifier state as the target in a transversal CNOT gate, after which the verifier is measured in the  $Z$  basis. In a CSS code, the target  $|\bar{0}\rangle$  will be in an equal superposition of codewords of the underlying classical code, which form a closed group under binary addition. When applying the CNOT gate, the lost qubits will behave as though in state  $|0\rangle$ . Thus the state of both ancilla and verifier after interaction will be.

$$p_{00}|\psi_{00}\rangle\langle\psi_{00}|_a \otimes |\bar{0}\rangle\langle\bar{0}|_v + \sum_{ij \neq 00} p_{ij}|\psi_{ij}\rangle\langle\psi_{ij}|_a X^i_A X^j_B |\bar{0}\rangle\langle\bar{0}|_v X^i_A X^j_B, \quad (8)$$

i.e., the only error-free support for the verifier state will correlate to ancilla state  $|\psi_{00}\rangle$ , and hence successfully passing verification will project the ancilla into state  $|\psi_{00}\rangle$ . The resultant ancilla state, including the two lost qubits, will behave, when used as a logical CNOT source, like a superposition of all codewords where qubits  $A$  and  $B$  are in state  $|00\rangle$ , and hence will correctly leave any logical CNOT target (such as a data block) unchanged. The behaviour is equivalent in this respect to the case where qubits  $A$  and  $B$  were simply fully dephased (i.e. each underwent  $Z$  errors with independent probabilities  $1/2$ ). As in the  $[[7,1,3]]$  case, a subsequent ancilla measurement in the  $X$  basis will give an erroneous syndrome (which, however, does not violate fault-tolerance for distance-3). An equivalent process occurs when verifying against  $Z$  errors: an ancilla with lost qubits will be projected into a superposition of codewords corresponding to the lost qubits being in state  $|+\rangle$ , and hence the resultant ancilla will leave a logical CNOT source unchanged when used as a target. The consequence is that using standard verification techniques one can “remove”  $X$  or  $Z$  errors from an ancilla with lost qubits through postselection (in the sense that the postselected ancilla will not transfer such errors on to the data), but not both  $X$  and  $Z$  errors.

However, such a process will not remove the loss errors. Thus, in the Knill technique, if the corresponding ancilla with losses passes verification, the output data state will have multiple loss errors. To remove this possibility, one can simply apply LRUs on each qubit of the  $|\bar{0}\rangle$  state after it is created. No further LRUs are required as part of the QEC procedure. Hence all three ancilla techniques can be made fault-tolerant against losses in the  $[[7,1,3]]$  code by adding only 7 additional LRU operations, compared to 28 (Knill technique, employing post-encoding LRUs on both ancillas and verifiers) or 35 (Steane technique, as with Knill plus 7 LRUs on the data) in [1].

#### IV. HIGHER DISTANCE CODES

A complete analysis of fault-tolerance in higher-distance codes requires verifying that, for a code of distance  $d$  and correctable number of errors  $t = (d-1)/2$ , that  $l < t$  gate failures in the QEC produce no more than  $l$  errors on the data block. In general, this can be quite complex and require multiple rounds of verification. We consider here the case of the Steane ancilla technique in higher-distance CSS codes, and specifically the example of the  $[[23,1,7]]$  Golay code, for which an ancilla verification procedure (from [11]) is shown in Figure 7

In general, such a technique will not be fault-tolerant against loss errors in our model. Unlike with distance-3 codes, where errors that only affect the syndrome measurement cannot violate fault-tolerance (since these result in at most one error on the data), a single gate failure can result in multiple syndrome-measurement errors and consequently multiple data errors. Similarly, since we previously only needed to demonstrate that single failures were tolerable, it was sufficient to note that any pattern of multiple errors on an ancilla would be detected by an error-free verification. At higher distances one must also consider the case of errors on both the ancilla and verifier, and the effect of different error patterns on the ancilla.

As discussed above, in the presence of loss, verification can still be used to prevent the transfer of  $X$  or  $Z$  errors to the data from a single ancilla, but not (without additional modification) both. In the common higher-distance case where multiple ancilla errors can lead to multiple data errors due to an incorrect syndrome, the possibility of an ancilla created with multiple losses from a single gate failure would only be compatible with fault-tolerance if such ancillas were always removed in verification. But, in general, an effective superposition of  $Z$ -basis codewords (i.e., a combination of lost and non-lost qubits which behaves like such a superposition when used as a CNOT source) will have a non-zero probability to pass verification against  $Z$  errors. Thus there is a non-zero probability for an ancilla with multiple losses to pass multiple verification rounds (and lead to multiple data errors through the syndrome), even in the absence

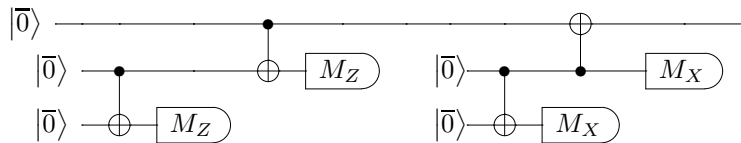


FIG. 7. Verification of a  $|\bar{0}\rangle$  ancilla in the  $[[23, 1, 7]]$  Golay code

of additional gate errors, and fault-tolerance is lost in general.

We note, however, that when our loss model applies this does considerably simplify the analysis of those *verifier* states which, as is common, only undergo a single interaction with the ancilla, even if the ancilla may interact with multiple verifiers as well as the data. This is because in many ancilla creation circuits a given qubit will consistently be used as either always the source, or always the target, of a CNOT gate. If, for example, a verifier state is being used to test for  $X$  errors, and hence interacts with the ancilla as a CNOT target, and then is measured in the  $Z$  basis, a loss error on one of these “always target” qubits is equivalent (or, depending on how losses are treated in measurement, no worse) in behaviour to replacing the qubit at the point of loss with a qubit in state  $|+\rangle$ . The same is true for losses of “always source” qubits. Thus any such error can be treated as a standard single-qubit error for the purpose of fault-tolerance analysis. (We note that the  $[[7, 1, 3]]$  Steane code is a special case of this: as discussed above, this QEC is fault-tolerant for any set of Pauli errors on the ancilla or verifier alone, thus we only need consider the effect of loss for operations after these states are created. For higher-distance codes we need to consider the possibility of errors on both, hence we must consider the specific behaviour of losses within the ancilla/verifier creation circuits and the distinction between “always source” and “always target” qubits).

More generally (assuming CNOT is the only two-qubit gate) we can use our notion of  $|0\rangle/|+\rangle$  replacement to represent any loss error occurring to qubit  $q$  at point  $P$  as the combination of the following:

- A single-qubit error occurring immediately before every CNOT gate applied to  $q$  after  $P$ , unless immediately following a CNOT gate where  $q$  acts in the same role (control or target).
- A single-qubit error occurring before any measurement on  $q$ , unless that measurement is in the  $X/Z$  basis immediately following  $q$  acting as a CNOT source/target respectively.

Such a mapping will, in general, represent a single loss error as multiple non-loss single-qubit errors, and hence will demonstrate fault-tolerance against a smaller num-

ber of loss errors than non-loss errors. However, it provides a lower bound on the number of errors which are tolerable, higher than that which an analysis of more general loss models would require.

## V. CONCLUSIONS

We have demonstrated that, for a model of qubit leakage applicable to many practical two-qubit gates, especially in optical applications, standard QEC techniques for distance-3 CSS codes can be made fault-tolerant with very few additional operations, leading to significant resource savings over the circuits required for more general loss models. Standard ancilla verification techniques, when passed, can effectively “project” lost qubits into states which do not transfer multiple errors to the data, because the non-lost qubits are projected into a state compatible with the behaviour of the lost qubits. While we have focused on photonic computing to motivate our loss model, we believe the same model may be applicable to trapped-ion quantum computation, which is also subject to leakage errors [12], and where the precise tuning of driving pulses to atomic transitions to perform two-qubit interactions may leave qubits unchanged if interacting with leaked qubits, as required by our model.

While higher-distance codes will generally require more additional operations (although still fewer than a more general loss model would require), the same principles allow the analysis of fault-tolerance to be considerably simplified compared to more general models.

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